Coulomb Integrals

Physics 204B CSU Chico
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1 Physical Integrals

Integrals have a bad rap. I blame Calc II. It was a tough class and a lot of it is lost, if not for being legitimately difficult, than for seemingly endless procedures with lack of application. In these notes I will try to walk you through what we need to know about integrals for 204B, i.e. the set up. The doing of the integral is "just math" (although in many cases it will require a trig sub that is harder than the original problem!) I have resisted the urge to expand on ideas as much as possible to be as clear and concise as possible.

Ok so what IS an integral? For these notes and this topic I will ask you to suspend your notion of integrals having anything to do with areas under curves. This is a *model*. It gives a nice visual when first learning about integrals and for some physical applications ($\int$ Impulse dt = momentum, $\int$ Force ds = work are usually presented this way) but will be a little counter intuitive for how I will be presenting.

Use the integrals-are-addition approach for these type of problem. Integrals are a special class of a summation (think Σ). You have no doubt seen this logic in Calc I (Riemann sum) but with an "area under the curve" twist. We are going to diverge from that logic and think of integrals purely as adding machines. Somehow we were gifted with a method of adding an *infinite* number of *infinitely* small things (and we complain about trig subs!)

Lets get into some examples

2 Coulomb Integrals

When first encountering a Coulomb integral it may seem overwhelming. Don't panic. There are many ways to set these up and they all give the correct answer. You invent the coordinate system, nature does not care what you use (just be sure to make your choice of coordinates explicit so you don't confuse the person grading your work, or worse, confuse yourself!) Break the problem down into the parts we ALWAYS need:

- dq, a differential chunk of charge. We always need to get from dq to d(object) and then to d(variable) so we can do the integration. We usually do that with a density (more on that later)
- r, the scalar distance from dq to P, the point we want the field at
- $\hat{r}$, The unit vector that gives the direction from dq to P

See if you can identify these in figure 1.

2.1 Find the field of a rod of uniform charge density on axis

We are given a rod of charge of length L and charge Q. We are asked to find the field at a point P a distance D away from the FAR end of the rod.

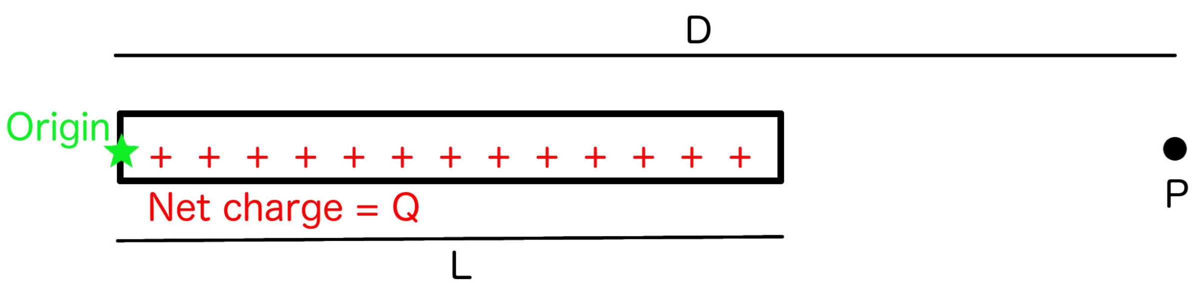


Figure 1: What the question you are given would look like

Start from Coulombs law for a small piece of charge, dq

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r} \tag{1}$$

We can see where the integration-is-addition state of mind pays off. The total field is just the sum of the fields due to little charges (dqs). We need to unpack our three objects of interest ($dq, r, \hat{r}$) and get them all in terms of a single variable. Then we do the integral on this variable.

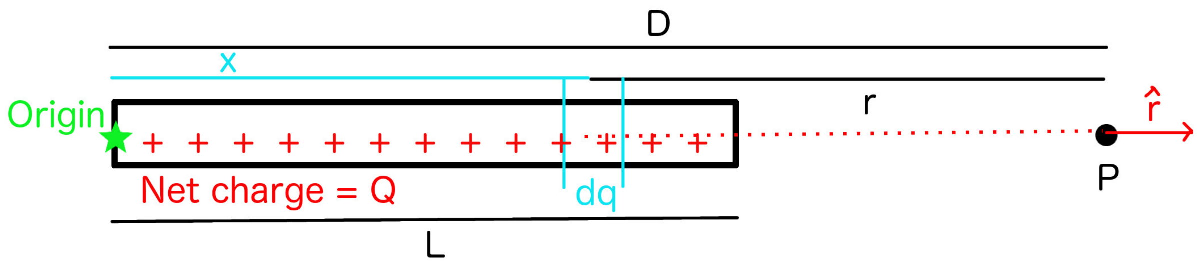


Figure 2: What your set up might look like

Be warned; we are getting into the meat of these notes. Read carefully!

dq We need to decide how we are going to break up the object into little d(object)s and then get to d(variable). This is the hardest part for most people. This example has the nice property of being in a neat line so we will take advantage of that and integrate over a variable that I'll call x. That means a position on my object can be specified with a single variable, x. I'll choose x as the distance from the origin to d(object). Also notice that d(object) becomes dx.

In order to get from dq to dx we need to think about how much charge is in a small length of the rod. An example that seems to help is thinking about it with mass. I start with a uniform stick that weighs 1 kg (M) and is 1 meter (L). If I gave you $\frac{1}{10}$ of that stick ($l = .1$ meter) you would have no trouble telling me how much mass you had (m). Think about what you did behind the scenes. 1 meter of stick is 1 kg, so the DENSITY is $1\frac{\text{kg}}{\text{meter}}$. To get the mass of the chunk you multiply the length of the chunk by the density. We could say that

$$m = \frac{M}{L}l \tag{2}$$

Now just swap M with Q, m with dq, and l with dx. We get

$$dq = \frac{Q}{L}dx$$

or

$$dq = \lambda dx$$

where $\lambda \equiv \frac{Q}{L}$

Another equivalent way to think about this is to take a ratio, $\frac{l}{L}$ (how much of the total length is the chunk) and multiply that by the total mass. Thinking about it this way we get

$$m = \frac{l}{L}M \tag{3}$$

These two are algebraically equivalent but the two forms represent different ways of thinking about it.

r The other two are simple after that! r is simply the distance from dq to P. Since we took the time to get dq in terms of dx, and we know dx is a distance x from what I called my origin,

$$r = D - x$$

$\hat{\boldsymbol{r}}$ in this case we hardly need to think about $\hat{r}$. Since every vector that points from each dx to P is in the same direction, $\hat{r}$ is a constant, and we can pull it out of the integral we are about to do. We can obviously say

$$\hat{r} = \hat{r}$$

Every time I do a Coulomb integral it starts with clearly writing dq, r and $\hat{r}$ at the top of my paper. The next step will show why. Eq. 1 becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\hat{r}\int_0^L \frac{dx\lambda}{(D-x)^2} \tag{4}$$

or in terms of the given quantities

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{L}\hat{r}\int_0^L \frac{dx}{(D-x)^2}$$

And we are done! You should be able to recognize this as a u sub, but that is not the important part and I will leave you to solve it, check units etc.

One important note. It is important to set the bounds from small to big. You will be off by a minus sign if you don't. Luckily this is easy to check. Just make sure the field points away from positive charge.

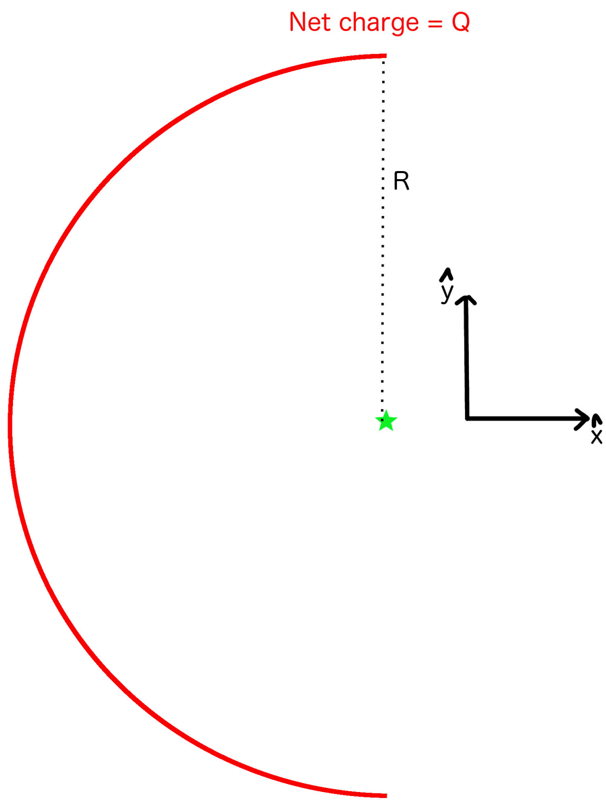
2.2 Find the field of a rod of non-uniform charge density on axis

This hardly deserves its own section because we were so careful in the first example. If you are given a non-uniform charge density do the problem the exact same way, with the exception of using the given density instead of forming one on our own. Lets say we were given the same problem, but we don't know the total charge, Q. Instead we are given $\lambda(x) = \frac{q_0}{l_0}\sin(2\pi\frac{x}{L})$, where q_0 and l_0 are constants (we need them to make the units work). All we do is drop this in to eq. 4. Since λ is not constant, we can't pull all of it out of the integral.

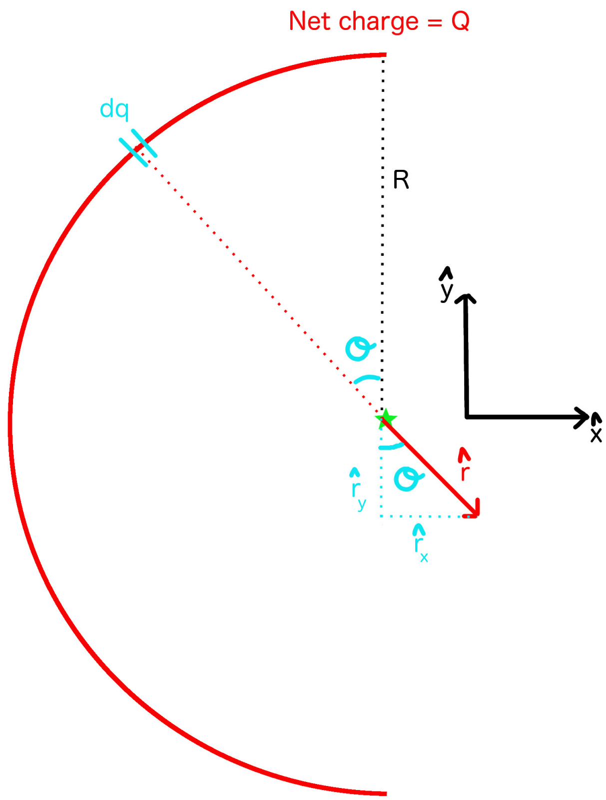
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q_0}{l_0}\hat{r}\int_0^L \frac{\sin(2\pi\frac{x}{L})}{(D-x)^2}dx$$

This integral is a little harder, but remember, you won't be asked to solve it!

2.3 Find the field of a semicircle at the center



This presents a different challenge in that the contributions to the field, $d\vec{E}$, are in more than one direction. We can't get away with simply saying the field is in the $\hat{r}$ direction. This will complicate $\hat{r}$. See if you can do it before moving on.



dq This time I will make my variable of integration θ, the angle from the vertical. There is no rule that I had to do that. If you can think of a way to specify a unique chunk of charge, dq, you can use it. This could be the height or some length element on the rod. Or you could measure the angle from the horizontal and go from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$. It is all fair game.

So we need to get from dq to $d\theta$. To get the density we need the total length and total charge. Remember that the length of an ark is some angle (measured in radians!) times the radius of the circle. We get a few relations we need from this statement

$$L = \theta_{total} R = \pi R$$

$$dl = d\theta R$$

Again, the density, λ, is $\frac{Q}{L}$ so

$$dq = \lambda dl$$

or

$$dq = \frac{Q}{\pi} d\theta$$

Where I canceled R

Now just like in the first example there is another way to think about this. We form a ratio of how much of the (semi)circle is in our chunk to the total circle. Then multiply this ratio by Q. This is analogous to eq. 3.

$$dq = \frac{d\theta}{\pi} Q$$

This ended being simpler, but again don't stress about doing it the "right way".

r r is constant for the whole semicircle. This one is easy!

$$r = R$$

$\boldsymbol{\hat{r}}$ $\hat{r}$ is not constant like it was in the first example. It has **two** components and we need to treat them independently when we integrate. That means **two** integrals. Now it turns out one of them will come out zero because of symmetry and we can kill it off when we define $\hat{r}$ but lets pretend we didn't know that.

$$\hat{r} = \sin\theta\hat{x} + \cos\theta\hat{y}$$

By looking at the picture, you should be able to see that the net field, $\vec{E}$, is in $\hat{x}$. You would be correct. We can drop the whole $\hat{y}$ term before we start setting up integrals. I encourage you to do this... Once you know why it works.

Since we did all the work up front, all that is left to do is shove the three things into eq. 1.

After simplifying we get

$$d\vec{E} = \frac{Q}{4\pi^2\epsilon_0 R^2} d\theta \Big[\sin\theta\hat{x} + \cos\theta\hat{y}\Big]$$

so

$$\vec{E} = \frac{Q}{4\pi^2\epsilon_0 R^2} \Big[\hat{x}\int_0^\pi \sin\theta d\theta + \hat{y}\int_0^\pi \cos\theta d\theta\Big]$$

A nice way to look at this is to imagine the integrals returning pure numbers that are acting as weighting factors for the unit vectors. Picture changing the bounds stretching, squeezing and rotating the resulting field vector. Since the second integral is zero, $\hat{y}$ gets zero weight. We are left with a field only in $\hat{x}$ as advertised.

$$\vec{E} = \frac{Q}{4\pi^2\epsilon_0 R^2}\hat{x}\int_0^\pi \sin\theta d\theta$$