Measurement of the Gravitational Constant with the Cavendish experiment

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Abstract

Between 1788 and 1789 Henry Cavendish (1731 – 1810) effectively weighed the earth using a torsion balance[1]. The experiment was devised by geologist John Michell in 1783. While originally thought of in terms of finding the density of the earth (or specific gravity) the modern experiment is cast in terms of finding the constant of proportionally, $G$, in the Newtonian law of universal gravitation,

$$\vec{F} = G \frac{m_1 m_2}{r^2} \hat{r}$$

Using a modern version of the torsion balance, we were able to measure $G$ as $5.9 \pm 0.3 \times 10^{-11} \frac{m^3}{sec^2 kg}$. However after correcting for systematic error we got $6.6 \pm 0.3 \times 10^{-11} \frac{m^3}{sec^2 kg}$, placing the known value of $6.674 \times 10^{-11} \frac{m^3}{sec^2 kg}$ well within our margin of error.

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I. INTRODUCTION

Gravity is an extremely weak force compared to other fundamental forces. Any effect it has between daily objects is completely overpowered by the earth’s gravity or other forces. In our experiment, the force of gravity between the large and small lead balls can be easily calculated once we know $G$ and is on the order of $10^{-7}$ times less than that between the small ball and the earth. This ingenious apparatus can detect incredibly small forces.

The torsion balance uses a very thin tungsten wire (about $\frac{1}{1000}$ of an inch thick) to support the dumbbell as seen in Figure 1. We shift large lead balls from one side of the dumbbell to the other (Figure 2). By observing the reflection of a laser off of an attached mirror onto a scale (Figures 3 and 4), we can calculate the restoring torque of the wire and the angular deflection of the dumbbell. From these and a few other easily measured quantities we can calculate $G$.

A. Theory

For simplicity we will neglect smaller effects at first and then introduce them in the next section. We begin by considering the dumbbell in equilibrium where the sum of torques is
FIG. 2. Torsion balance and laser for tracking

FIG. 3. \( \Delta \theta \) is half of what we expect. The angle of incidence is equal to the angle of reflection, so as the mirror (and therefore the normal) moves a \( \Delta \theta \), both of these components contribute a factor of \( \Delta \theta \). Again, angles are exaggerated to show geometry.
zero (Neglecting $F'$ for now as seen in 5)

$$-k\theta + 2G\frac{Mm}{l} = 0 \tag{1}$$

or

$$k\theta = G\frac{Mml}{b^2} \tag{2}$$

where $k$ is the spring constant for the torsion spring (in units of force * length, instead of force * length$^{-1}$ as would be for a linear spring), $M$ and $m$ are the large and small masses respectively, and $b$ is the distance between the centers of the balls.

Next consider shifting the large balls to opposite sides. After the dumbbell settles to it's new equilibrium Equation 1 becomes

$$k\theta - G\frac{Mml}{b^2} = 0 \tag{3}$$

which reduces to Equation 2.

After reducing Equation 3 to 2, adding Equations 3 and 2 yields

$$\Delta \theta \equiv 2\theta = 2G\frac{Mml}{kb^2}$$

or

$$G = \frac{\Delta \theta}{2lM}$$

$$\frac{b^2}{m}$$  

$$\frac{k}{2}$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$
In practice we shift the balls before equilibrium is reached, inferring the required quantities from the oscillatory motion. We can only measure a difference in angles, so it is preferable to get equations in terms of $\Delta \theta$.

$\Delta \theta$ can be related to the distance we measure on the scale by

$$\Delta \theta = \frac{1}{2} \arctan \left( \frac{\Delta s}{L} \right) \approx \frac{1}{2} \frac{\Delta s}{L}$$

where the factor of $\frac{1}{2}$ can be seen on figure 3, $L$ is the distance from the mirror to the scale, and $\Delta s$ is the difference in equilibrium positions.

From Equation 4 we can eliminate $k$ and $m$ in favor of the period of oscillation, $T$, the displacement, $\Delta s$, and the damping time $\beta$ (though this has a very small effect and will not be introduced until the next section).

Approximating the rotational inertia as two point masses both $\frac{l}{2}$ away from the axis of rotation we have

$$I = 2m \left( \frac{l}{2} \right)^2 = ml^2$$

So
\[
\frac{k}{m} = \frac{k l^2}{I/2} = \frac{\omega_0^2 l^2}{2} = \left(\frac{2\pi}{T}\right)^2 l^2
\]

We find that
\[
G = \frac{b^2 l \Delta s}{8ML} \left(\frac{2\pi}{T}\right)^2
\]

B. Experiment

The dimensions of the torsion balance were measured carefully. The largest source of error comes from \( b \), the distance between the centers of the large and small balls and is the sum of the radius of the large ball and half of the width of the chamber. We are assuming that the small ball is at the center of the chamber at all times. This simplification is justified because the balls only move about \( .005 \text{rad} \ l/2 = .02 \text{mm} \). We also must assume that we lined everything up correctly and the equilibrium position \textit{without} the large balls places the small balls exactly halfway in the chamber. We estimate \( \delta b = 1 \text{mm} \) so \( b = .047 \pm .001 \text{m} \)

\( l \), the distance between the centers of the small balls, can be easily measured with vernier calipers. Measuring between the insides of the small balls and the outsides and subtracting off two times the radius of the small balls we have \( l = .0952 \pm .0005 \text{m} \)

\( M \) was found simply by weighing the large balls. \( M = 1.49 \pm .01 \text{kg} \)

\( \Delta s \) and the period, \( T \), were found with the main procedure.

The torsion balance was set up and a laser was aligned to hit a scale located a distance \( L = 6.15 \pm .03 \text{m} \) from the mirror (Figure 4). The glass coverings were wrapped in aluminum foil and everything was grounded to prevent any electrostatic forces from interfering (Figure 2). Everything was carefully leveled and the dumbbell centred. A camera and intervalometer were used to take photos every 5 seconds. This data was fed into Tracker software which generated an array of positions that was exported into Python for analysis (Figure 6). By curve fitting a decaying exponential \(^1\) to each run we were able to get 5 values for \( T, \beta, \) and \( s_{eq} \). We are interested in differences in \( s_{eq}, \Delta s \) which we only had four values for. Tracker was set to follow the entire \( \approx 1 \text{cm} \) spot, but had a tendency to drift to the brighter side. This was solved by having it look only at a small portion of the large spot. This generated noise as it bounced around within the spot from frame to frame, but it seemed

\(^1\) We used \( Ae^{-\beta t} \sin(\omega t + \phi) + s_{eq} \) but had no use for \( A \) or \( \phi \)
to be equally distributed within the bright portion of the spot. By having several runs with more frequent data points than others with similar apparatuses \([2] [3]\), we are confident that that there is relatively low random error and that the reduction in systematic error caused by a drifting spot is much worse. We estimated the uncertainty using the standard deviation between runs and the spatial resolution of Tracker, 1mm. Temporal error was negligible as the NIST clock seen in the photo did not drift over the entire 2 hour run. We measured \(T = 353 \pm 0.07\) sec, \(\Delta s = 0.066 \pm 0.001\) m and \(\beta = 0.0014 \pm 0.0001\) sec\(^{-1}\).

The two largest sources of relative uncertainty come from \(b\), (2.1\%) and \(\Delta s\), 1.5\% \(^2\). Because \(b\) is squared in Equation 7, it dominates the uncertainty. All other measured quantities are < .1\%. Since the measurements were all independent they were added in quadrature and we neglected all but \(\delta b\) and \(\delta \Delta s\) in the calculation. We found

\[
G = 5.9 \pm 0.3 \times 10^{-11} \frac{m^3}{sec^2 kg}
\]

\(\frac{\delta \beta}{\beta}\) was quite high at 7\%, but never enters into the calculation. It is only used to show that the damping does not significantly effect the measured period
C. Corrections

Several approximations were made in the derivation of Equation 7 which we will either justify or account for, namely a small angle approximation in Equation 5, observed period of oscillation vs natural period (due to damping), rotational inertia, and neglecting the torque of opposing balls.

1. Small angle approximation

The small angle approximation of Equation 5 made use of the ratio $\Delta s$ to $L \approx .01$. 
$$\arctan(0.01) = .99997$$ which is well within the overall uncertainty.

2. Damping

Following Chapter 5 in Taylor [4] we find that 
$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

Where $\omega_1$ is the observed frequency and $\omega_0$ natural frequency (what we measure vs $\sqrt{k/I}$) and $\beta$ is the parameter that determines the time scale in the envelope, $e^{-\beta t}$.

$$\omega_0 \omega_1$$, then is $\sqrt{1 + (\frac{2}{\omega_1})^2} \approx 1.003$. $\frac{1}{3}\%$ is negligible as well.

3. Rotational inertia

We assumed that the rotational inertia was simply given by that of two point masses a distance $\frac{l}{2}$ away from their axis of rotation. Accounting for the balls extent in space (spheres with radius $r$) and the inertia of the aluminum rod we find the ratio of the new moment of inertia, $I'$ to the simplified $I$

$$\frac{I}{I'} = \frac{2m(\frac{l}{2})^2}{2m(\frac{l}{2})^2 + 2\frac{2}{5}mr^2 + \frac{1}{12}m_{rod}l^2} = \frac{1}{1 + \frac{2}{5}(\frac{2r}{l})^2 + \frac{1}{24} \frac{m_{rod}}{m_{ball}}}$$

where 
$$\frac{m_{rod}}{m_{ball}} = \frac{\frac{l}{2}r_{rod}\rho_{al}}{\frac{4}{3}r_{ball}\rho_{pb}}$$
We can easily measure or look up all of these quantities and find $I' = .98$. This tells us that the moment of inertia we used for the calculation was about 2% too small. Put another way, our physical moment was too big. We need to scale $I$ back by 2% to make it match our model. Multiplying $I$ in Equation 6 by .98 and watching it propagate through the calculations we see that it will have a net effect of increasing our value of $G$ by 4%. This is significant.

4. Torque from opposing balls

The final correction term comes from the attraction between opposing balls (see Figure 5). $\theta$ is greatly exaggerated, and the small balls are very close to being on the center line. The torque between a small ball and opposing large ball is given by $F'_\sin(\arctan(\frac{b}{\sqrt{b^2 + l^2}}))$ or $G \frac{mM}{(b^2 + l^2)^{3/2}}$. So just as we did for the inertia, we can say

$$\frac{F'}{F} = \frac{1}{(1 + \frac{l^2}{b^2})^{3/2}} = .085$$

That means that we can reduce our torque due to gravity in Equation 1 by 8.5%. This will propagate through and increase our measured value by another 8.5%, for a total of 12.5%.

The dominant error terms are still $b$ and $\Delta s$ so our random error remains relatively unchained. however our systematic error has been reduced greatly, accounting for the two largest sources.

Our new value for $G$ is given by

$$G_0 = 1.125 * G = 6.6 \pm .3 * 10^{-11} \frac{m^3}{sec^2 kg}$$ (8)