# Standing waves on a one string guitar

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## 1 Abstract

We built an apparatus and set of software that are very conducive to the study of physical waves on a guitar string. Preliminary tests do not show good agreement between expected and observed data, however this is likely to be corrected upon further investigation. There is a possibility that there is an flaw with the optical pickup design, however if this were the case we should not see such clear peaks of the harmonics. The likely issue is one of alignment or analysis.

## 2 Introduction

The initial conditions of a plucked guitar string are, to good approximation, a triangle whose apex is at $x = p$, $y = a$ (Figure 1). This is a simple peace-wise function (Eq. 5) and can be decomposed into Fourier components. By assuming a solution as a sum of the form

$$y(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) \tag{1}$$

we can produce an analytic solution [3]. Due to the rapid convergence of the series we only need a few terms to get a good looking solution (10 works great).

## 3 Analytic Solution

Wave equation with damping is

$$\frac{\partial^2 y}{\partial t^2} + 2\beta \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{2}$$

Where $\beta$ is a damping coefficient and $c$ is the wave speed.

We begin by nondimensionalization, noticing the equation has units of $[\frac{L}{T^2}]$. We have some freedom to choose out units. We could define wave speed in terms of mass density and tension but these are difficult to measure accurately. By defining the length of the string as the unit of length ($663 \pm 1mm$) and the fundamental frequency (varies with each run depending on tuning. The analysis script extracts it from the Fourier transform, and is around 85 Hz for the low E string and around 330 Hz for the high E) as the unit of frequency, we can derive all kinematic quantities [1]. Notice that now $c = 1$. Dividing through by $(f_1)^2 L$ we have a dimensionless equation. It's solution is [1][2]

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-\beta t} \big[ \cos(\sqrt{n^2 - \beta^2}\pi t) + \frac{\beta}{\sqrt{n^2 - \beta^2}} \sin(\sqrt{n^2 - \beta^2}\pi t) \big] \tag{3}$$

where

$$b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx \tag{4}$$

$f(x)$ is determined by the initial conditions in Figure 1:

$$f(x) = \begin{cases} \frac{ax}{p} & 0 \le x \le d \\ \frac{a(1-x)}{1-d} & d < x \le 1 \end{cases} \tag{5}$$

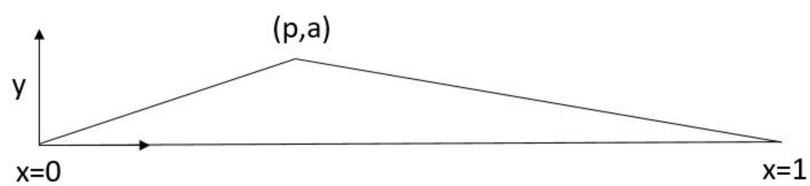


Figure 1: Initial conditions of a string about to be plucked.

This makes $b_n = \frac{2a}{(n\pi)^2(p-p^2)} \sin(np\pi)$

Equation (3) can be computed and plotted. The plots can be seen in figures 2, 3 and 4

[1] Where context makes it apparent, we leave off units of length and time if we are referring to these units

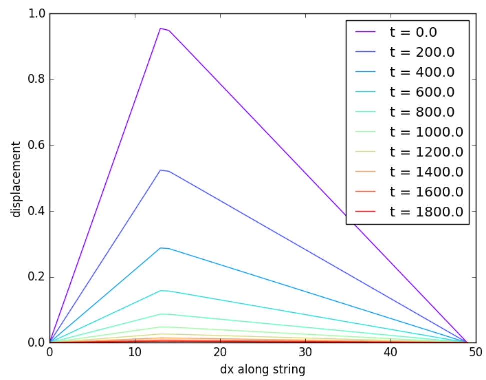


Figure 2: Damping was set to cause most motion to die out around t = 1800 (900 cycles) as seen in figure 8

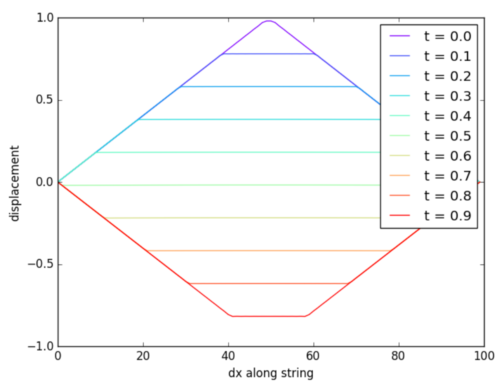


Figure 3:

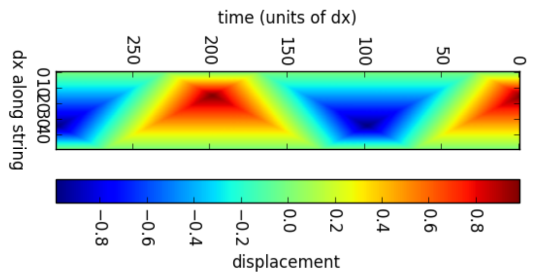


Figure 4:

# 4 Apparatus

The apparatus was constructed from a 2x1" block of walnut. Aluminum brackets were cut to size for use as a nut and bridge and slots were filed out for the string [2]. There was some buzzing that was resolved by placing bits of paper towel between the string and brackets. Optical pickups were made using surface mount led and phototransistors. The mount was designed in CAD and 3D printed (circuit figure 5.) The optical pickup design was chosen to allow very precise placement of the sensor as opposed to a magnetic pickup who would sense a longer section of string.

[2] Thanks to Jaydie Lee for help building and to the CSU Chico Art department for donating the wood

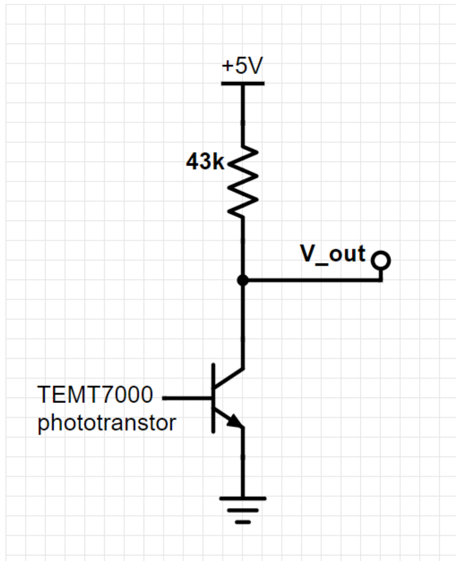


Figure 5: Optical pickup circuit

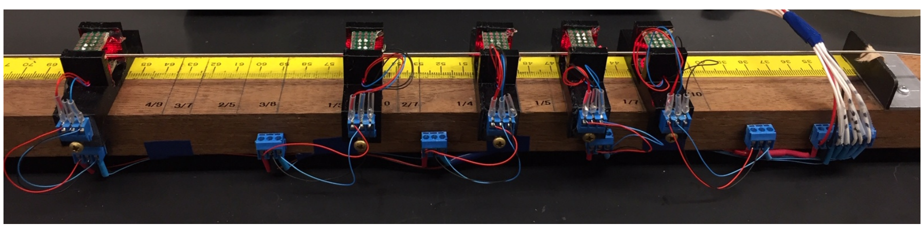

Figure 6: Side view of the "guitar."

## 5 Analysis

The analysis is interesting and speaks to the power of the Fourier transform. Looking at the signal up close has strange looking artifacts (figure 7), but the Fourier transform simply interprets them as periodic with the fundamental frequency and they don't effect the output.

The data is not bad, but very sensitive to the placement of the pickup. The pickups at higher order nodes do not display the expected elimination of modes, or even a noticeable dip in most cases. Using the analytic solution and calculating the Fourier transform at .5 makes all $f_{2n} = 0$. This was not observed (Figure 10). Estimating the uncertainty in the pickup location to be 2mm, translates into and uncertainty of .003 (remember, units of L are implied). Placing the "pickup" at .5 + uncertainty of .003 gives large dips in modes that were not observed. By analyzing with the "pickup" at .515 as seen

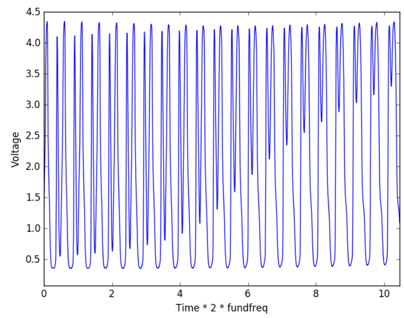


Figure 7: Strange artifacts ignored by Fourier transform

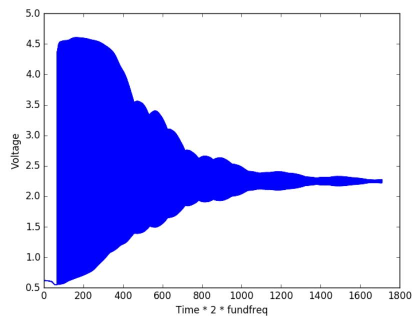


Figure 8: 2 time units = 1 period

in Figure 9 we observed similar results to the physical pickup at .5 ± .003. The pickups placed at higher modes require very high precision in their placement. Because the wavelength of the higher mode is shorter, that same millimeter of uncertainty represents a greater fractional uncertainty with respect to the shorter wavelength and hardly any dips are observed in the expected frequencies. This was not expected, as calculating the Fourier transform with the "pickups" within uncertainty gives very noticeable dips that were not observed in the data.

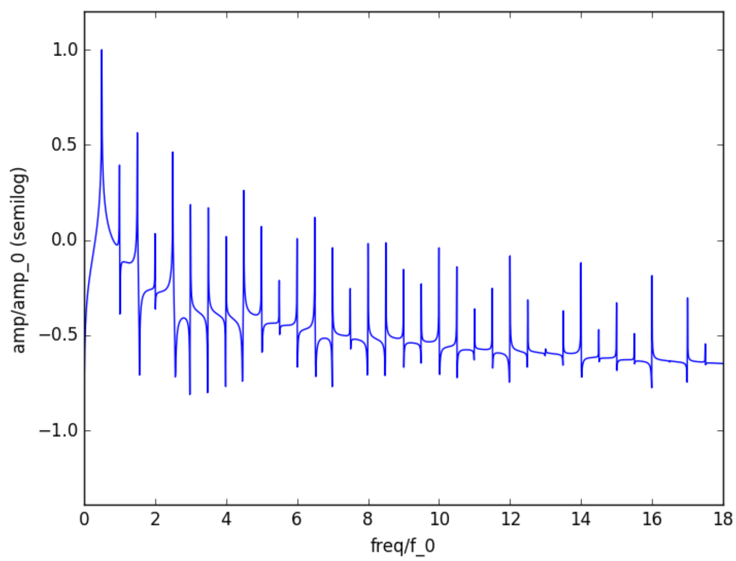


Figure 9: pickup at .515, pluck at .283

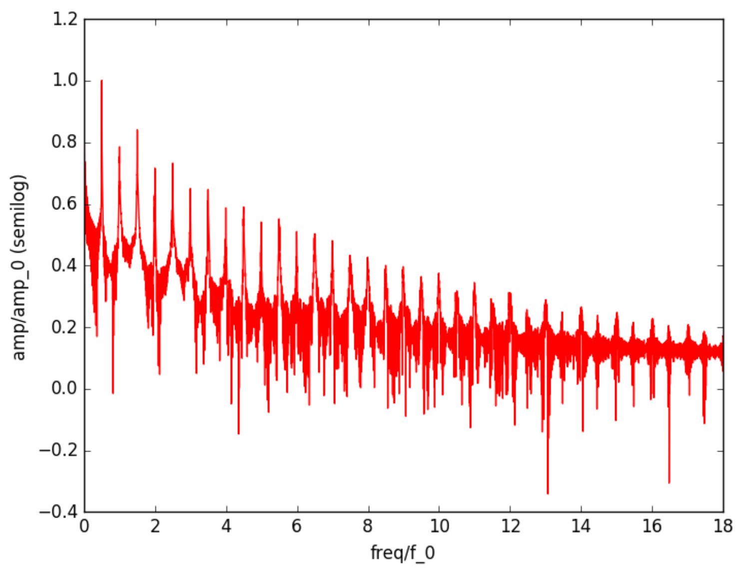


Figure 10: pickup at .5 $\pm\delta$ , pluck at .25 $\pm\delta$

## References

[1] Virginia Noonburg. *Lecture 16: More on the Wave Equation.* URL: http://uhaweb.hartford.edu/noonburg/m344lecture16.pdf. (accessed: 12.20.2018).

[2] Jason Pelc. *Solving the Sound of a Guitar String.* URL: http://large.stanford.edu/courses/2007/ph210/pelc2/. (accessed: 12.20.2018).

[3] James Ward Brown Ruel V. Churchill. *Fourier series and boundary value problems 3rd ed.* McGraw-Hill, 1978.