Portfolio Problem 2
Fourier Series

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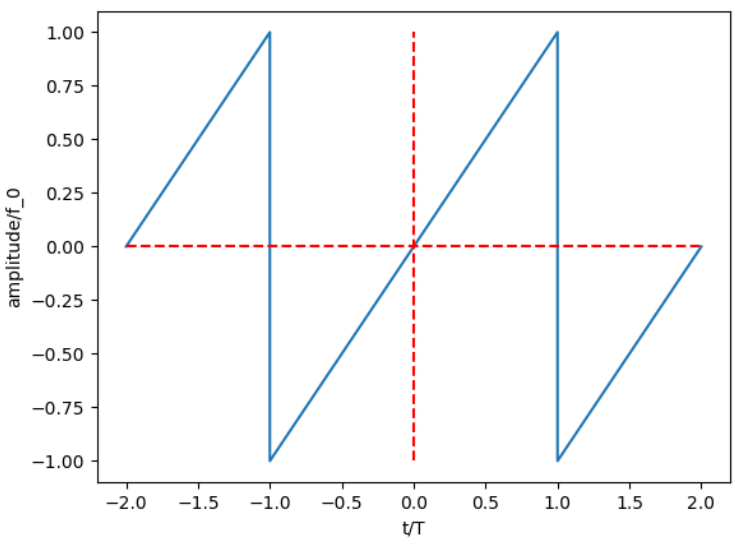
This problem has 2 main parts. The first is to take a given periodic drive force and represent it as a sum of sines and cosines. The next is to apply this sum of periodic forces to an oscillator with given parameters and see how it responds in the long term (after transients have died out).

1 Fourier Series: Breaking up the drive force

An oscillator is driven by a saw tooth function. This is a periodic function given by $f(t) = f_0 \frac{t}{T}$ where t is the time variable and T is half of the period.



The first step is to recognize that any periodic function can be written as a sum of elementary oscillatory functions; sines and cosines or (the real part of) complex exponentials. To do this we will need to convert our function to be in terms of π. This will standardize the period to be 2π and allow use of trig or complex exponentials. To do this we will map all t's to θ's with the relation

$$\frac{t}{T} = \frac{\theta}{\pi}$$

.

We can see that $\theta = \pi\frac{t}{T}$ which we express as

$$\theta = 2\pi\frac{t}{2T}$$

This can be thought of more clearly as

$$\theta = 2\pi\frac{1}{2}\frac{t}{T}$$

or θ is half of the ratio $\frac{t}{T}$ times 2π. If this is the argument of sine, when t=T, sine will have gone through half of a cycle, and this is just what we want. We now have

$$f(t) = f_0\frac{\theta}{\pi}$$

Where θ is a function of t.

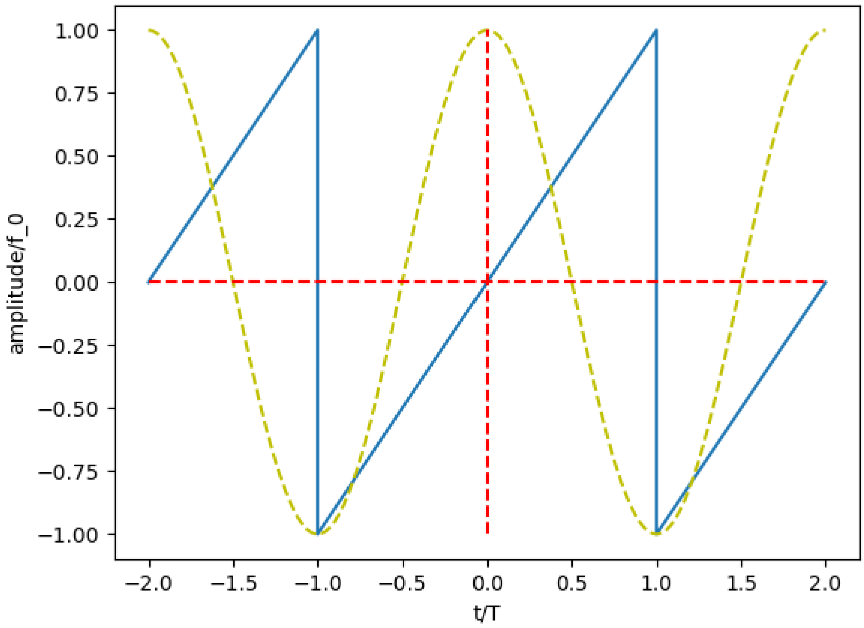
We now remember that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega t}$$

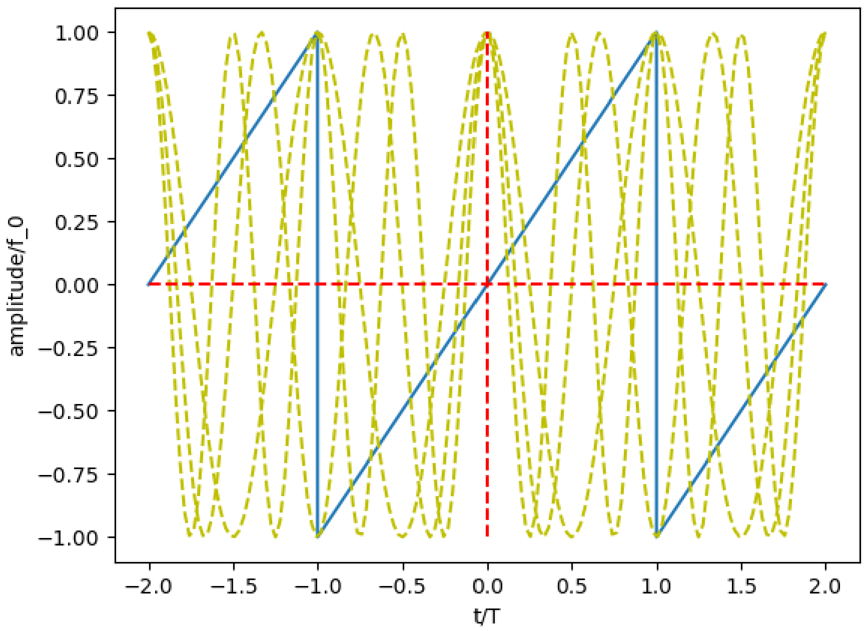
and

$$f(t) = \sum_{n=0}^{\infty} a_n cos(n\omega t) + b_n sin(n\omega t)$$

In this case we are given an odd function. We can see that if we were to sum cosines, the zeroth order term would add a constant that we don't want. The first order term would have the same period as the drive force, but the phase would be off.



This could be useful if we wanted to shift the phase of the function, but for this case we don't. For the second, third and fourth order terms of cosine we see that at t=0, it hits a max where we want a zero. These all add and it is clear that we don't want cosines in our sum

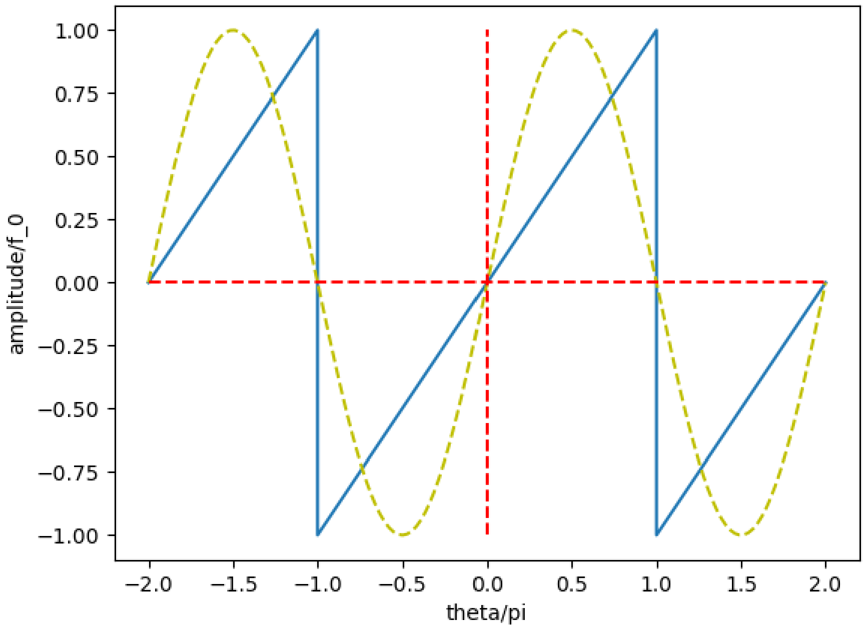


In fact this will happen for every order cosine and I will argue that all cosines can be thrown out and we can represent our function simply as a sum of sines.

$$f(t) = \sum_{n=0}^{\infty} b_n sin(n\omega t)$$

The zeroth order disappears, and like our function, sine of zero always returns zero. While they do not hit maximum at the same time, we trust that higher order terms can make up for this. The first order has the correct general shape.
While we are at it, lets replace the argument of sin with θ

$$f(t) = \sum_{n=1}^{\infty} b_n sin(n\theta)$$



The next task is to find b_n. This can be done with (one of) Fourier's trick(s).

$$\int_{onecycle} sin(n\theta) sin(m\theta) d\theta = \delta_{mn} \frac{cycle}{2}$$

Where δ is a kronecker delta. This can be simply shown two ways. If $n \neq m$, we have

$$\int_{onecycle} sin(n\theta) sin(m\theta) d\theta$$

using an angle sum identity we can simplify to a sum of two integrals which each evaluate to zero.
If $n = m$ then we have

$$\int_{onecycle} sin^2(m\theta)d\theta$$

which is $\frac{cycle}{2}$
Using this we can take our sum, multiply both sides by $sin(m\theta)$ and integrate over a cycle (ill go $-\pi$ to π).

$$\int_{-\pi}^{\pi} f(t)sin(m\theta)d\theta = \int_{-\pi}^{\pi} sin(m\theta)\sum_{n=1}^{\infty} b_n sin(n\theta)d\theta$$

Using Fourier's trick, the right hand side simplifies and we can make a substitution for $f(t)$ on the left that is specific to our sawtooth function. Technically this solves for b_m when $m = n$, but because they are equal at this point it is the same as writing b_n and replacing m with n so I will do that for continuity with the Taylor.

$$\frac{f_0}{\pi}\int_{-\pi}^{\pi} \theta sin(n\theta)d\theta = \frac{2\pi}{2}b_n$$

or

$$b_n = \frac{f_0}{\pi^2}\int_{-\pi}^{\pi} \theta sin(n\theta)d\theta$$

This is easily solved with 2 iterations of integration by parts. This is simplified with quick parts.

$\frac{d}{d\theta}$	$\int d\theta$
θ	$sin(n\theta)$
1	$-\frac{cos(n\theta)}{n}$
0	$-\frac{sin(n\theta)}{n^2}$

We get

$$b_n = \frac{f_0}{\pi^2}[-\theta\frac{cos(n\theta)}{n} + \frac{sin(n\theta)}{n^2} + \int 0 d\theta\Big|_{-\pi}^{\pi}$$

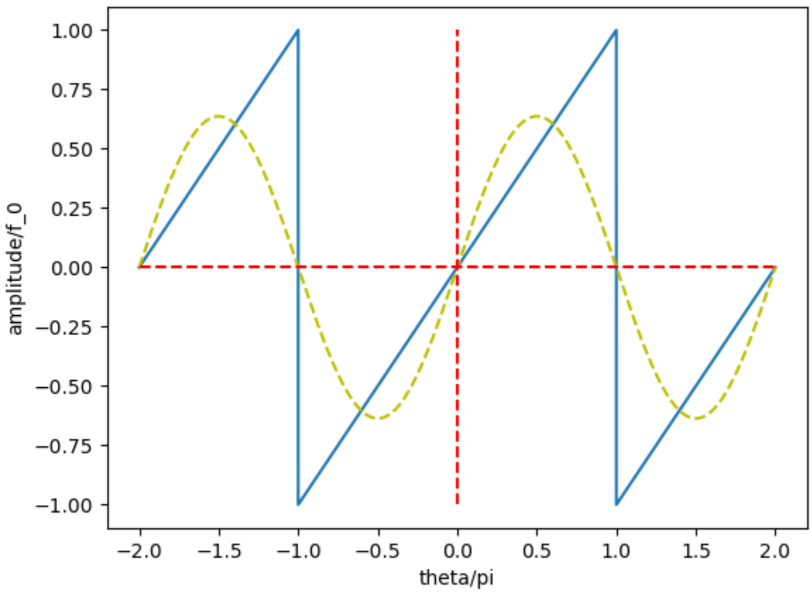
As n advances, $cos(n\pi)$ switches positive and negative. It evaluates to $(-1)^n$

$$b_n = \frac{f_0}{\pi^2}[-\pi\frac{-1^n}{n} + 0] - [\pi\frac{-1^n}{n} + 0]$$
$$b_n = \frac{f_0}{\pi}\frac{2}{n}(-1^{n+1})$$

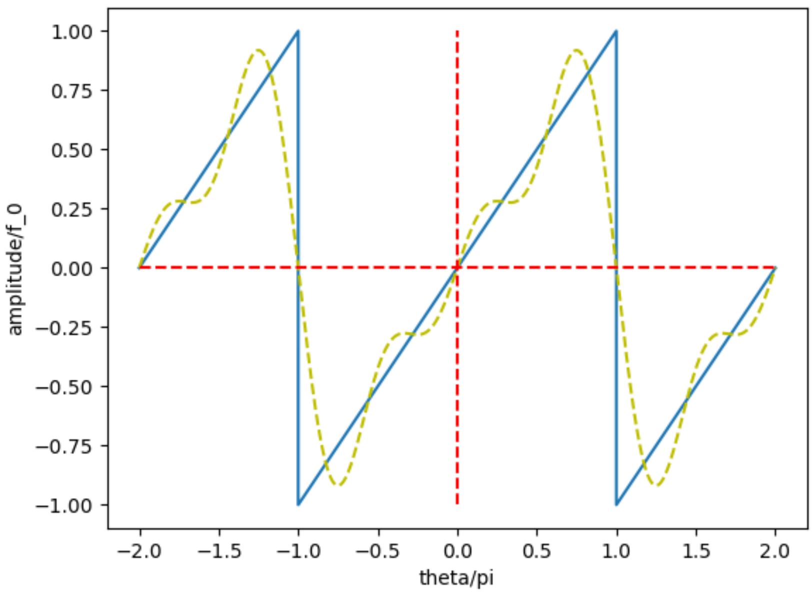
Where in the last step I combined the negative out front into the alternating term.

$$f(t) = \sum_{n=1}^{\infty} \frac{f_0}{\pi} \frac{2}{n} (-1^{n+1}) sin(n\theta)$$

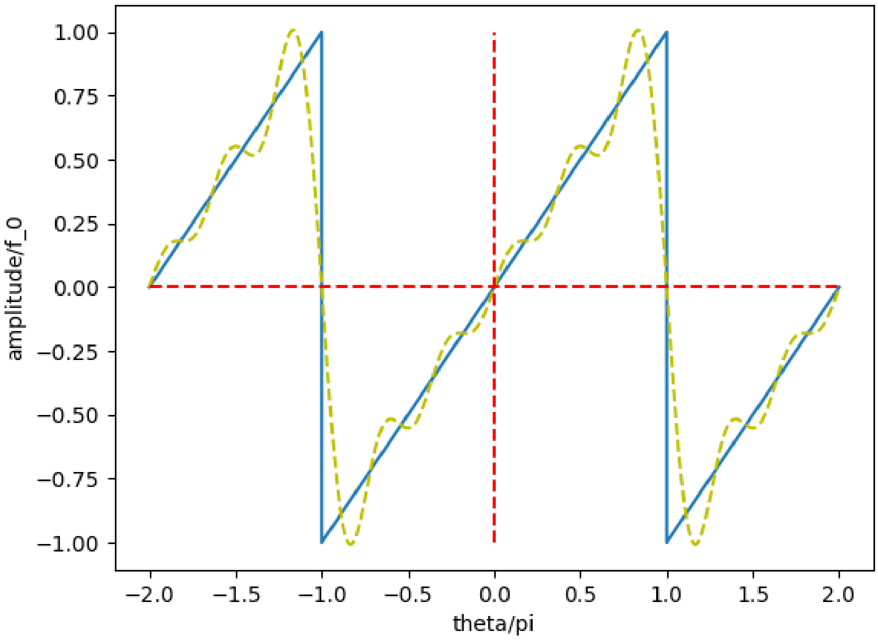
Converting this to python we can plot different combinations of resolution and n values. Here is $n = 1$



$n = 3$

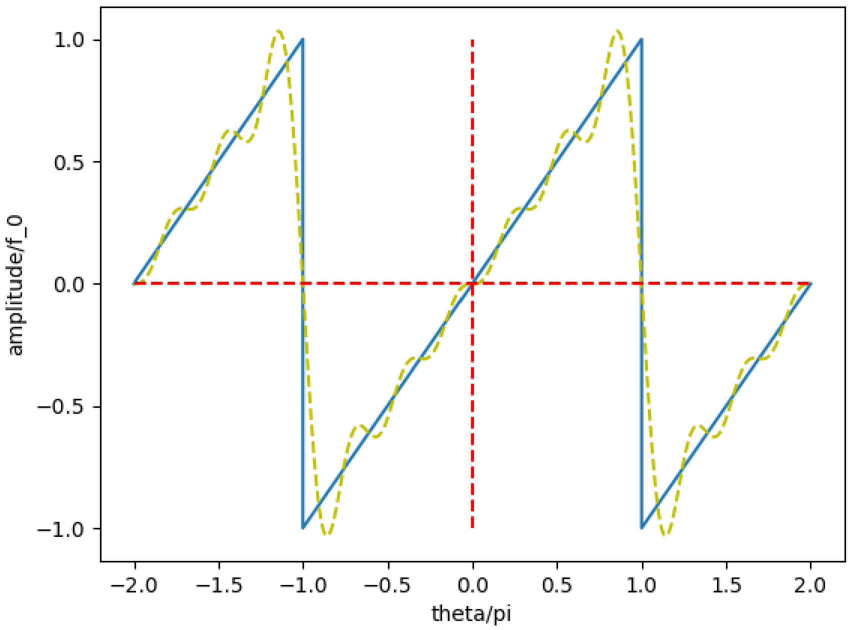


$n = 5$



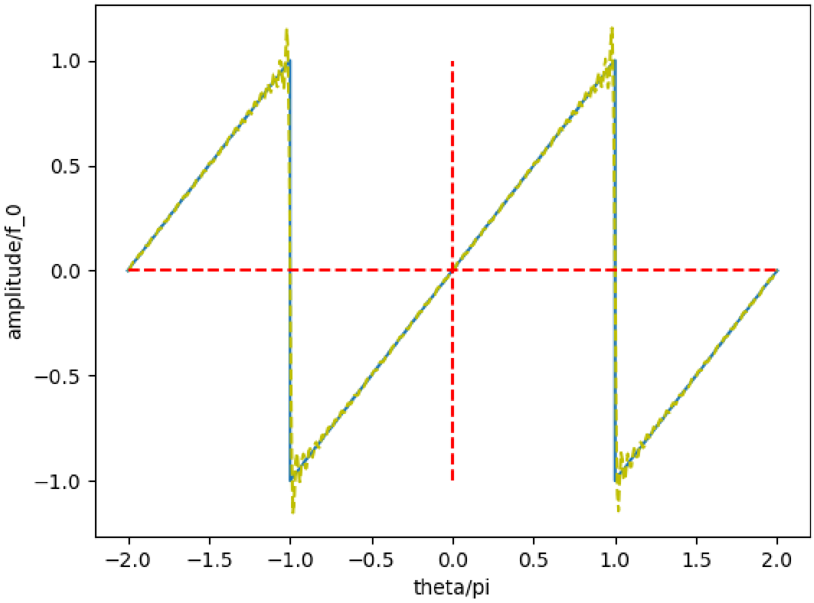
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
amplitude/f_0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
theta/pi

$n = 6$

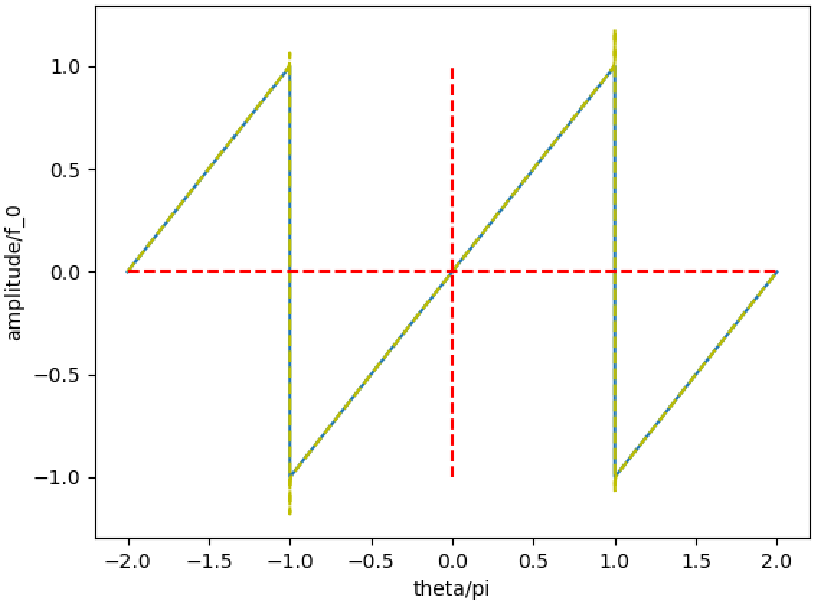


1.0
0.5
0.0
−0.5
−1.0
amplitude/f_0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
theta/pi

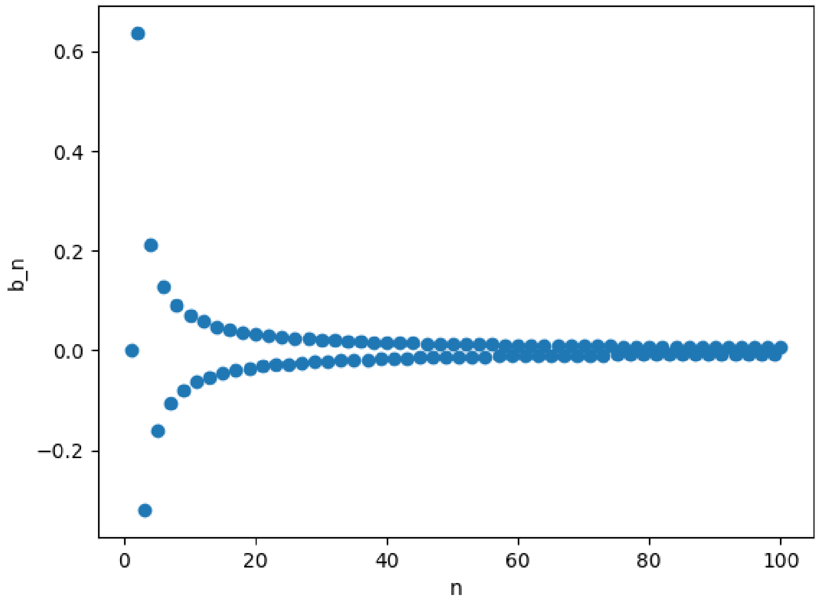
n=50



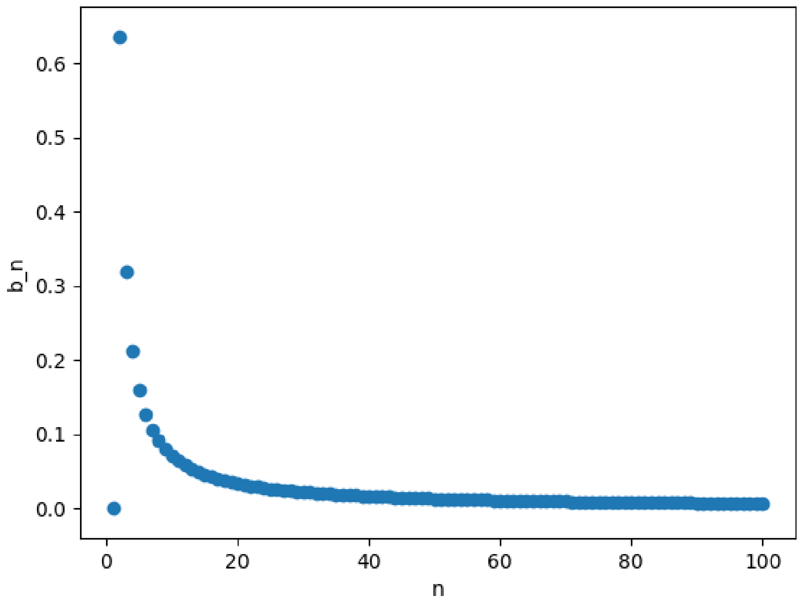
And of course, n=5000. I had to turn up the resolution quite a bit because there was so much error with this many terms. It took a while to run because my code is not very efficient



It is also interesting to notice the rapid convergence of the coefficients



which we can take the absolute value of to see the pattern more clearly



For this drive force each higher term contributes less and less. We will see this clearly in the next section.

With our drive force represented as a sum of sine and cosines we will now apply it to an oscillator and see the response.

2 RESPONSE

Starting from $f = ma$ we quickly arrive at

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{\text{periodic drive force}}{m} = \frac{f_0}{m} cos(wt) \tag{2.1}$$

next we will "make up" a new problem that will help solve our current problem.

$$i[\ddot{y} + 2\beta\dot{y} + \omega_0^2 y] = \frac{-f_0}{m} isin(\omega t) \tag{2.2}$$

adding (2.1) and (2.2) we get

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{f_0}{m} e^{-i\omega t} \tag{2.3}$$

Let $z_p = Ae^{-i\omega t}$ where A is an arbitrary constant and plug this in to our equation for z. Taking derivatives and canceling e's we get

$$-A\omega^2 - 2\beta\omega i A + A\omega_0^2 = \frac{f_0}{m} \tag{2.4}$$

To clean this up we introduce several new perimeters

$$\frac{\omega}{\omega_0} = \bar{\omega}$$

$$\tau = \frac{1}{2\beta}$$

$$Q = \tau\omega_0 \rightarrow \frac{\beta}{\omega_0} = \frac{1}{2Q}$$

Factoring out $A\omega_0^2$ from (2.4), applying our substations and solving for A we get

$$A = \frac{1}{1 - \bar{\omega}^2 - \frac{i\bar{\omega}}{Q}} \frac{f_0}{\omega_0 m}$$

Which of course goes into

$$z_p = \frac{1}{1-\bar{\omega}^2 - \frac{i\bar{\omega}}{Q}} \frac{f_0}{\omega_0 m} e^{-i\omega t}$$

This can be viewed in an interesting way;

$$z_p = (\frac{1}{1-\bar{\omega}^2 - \frac{i\bar{\omega}}{Q}} \frac{1}{\omega_0 m})(f_0 e^{-i\omega t})$$

The first term is a (complex) factor and the second is the drive. We really want x_p which is simply the real part of z_p. There are a number of ways to get this. I will multiply A by its complex conjugate and take the square root. Our drive force is made of sines, but we will need a phase term because response of an oscillator depends on how close the drive frequency is to resonance. With all of this in mind we can rephrase the problem in a Fourier way; that is

$$x_n = \frac{1}{m\omega_0^2} \frac{1}{\sqrt{(1-\bar{\omega}_n^2)^2 - (\frac{\bar{\omega_n}}{Q})^2}} b_n sin(n\omega t + \phi)$$

Where ϕ if the phase term and can be seen as the the arctangent of the imaginary part divided by the real part.
so we have

$$x = \sum_{n=1}^{\infty} \frac{1}{m\omega_0^2} \frac{1}{\sqrt{(1-\bar{\omega_n}^2)^2 - (\frac{\bar{\omega_n}}{Q})^2}} b_n sin(n\omega t - \phi_n)$$

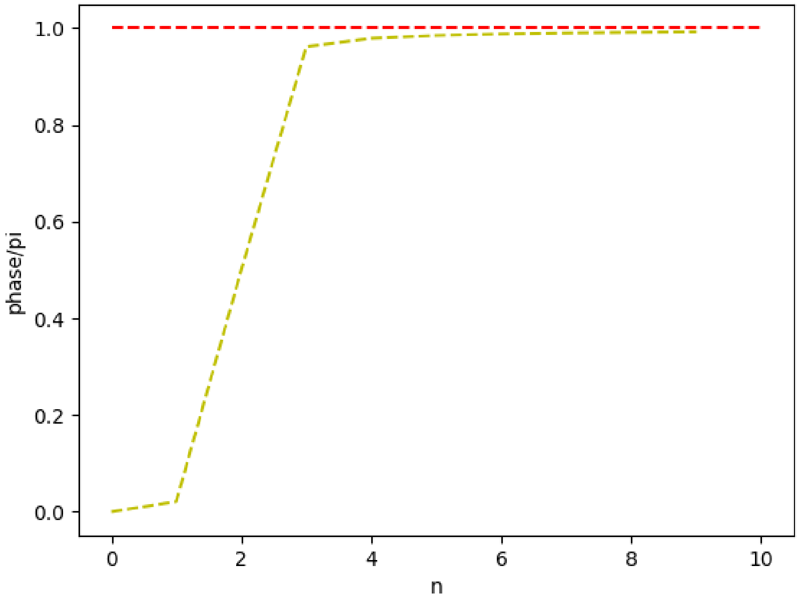
which I will scale (remembering b_n has f_0 in it)

$$\frac{x}{\frac{f_0}{m\omega_0^2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{(1-\bar{\omega_n}^2)^2 - (\frac{\bar{\omega_n}}{Q})^2}} (\frac{2}{\pi n}(-1^{n+1}))sin(n\omega t - \phi_n)$$

(for convenience the left side will be $\bar{x}$ from now on)
where

$$\phi_n = arctan(\frac{\frac{\bar{\omega_n}}{Q}}{1-\bar{\omega_n}^2})$$

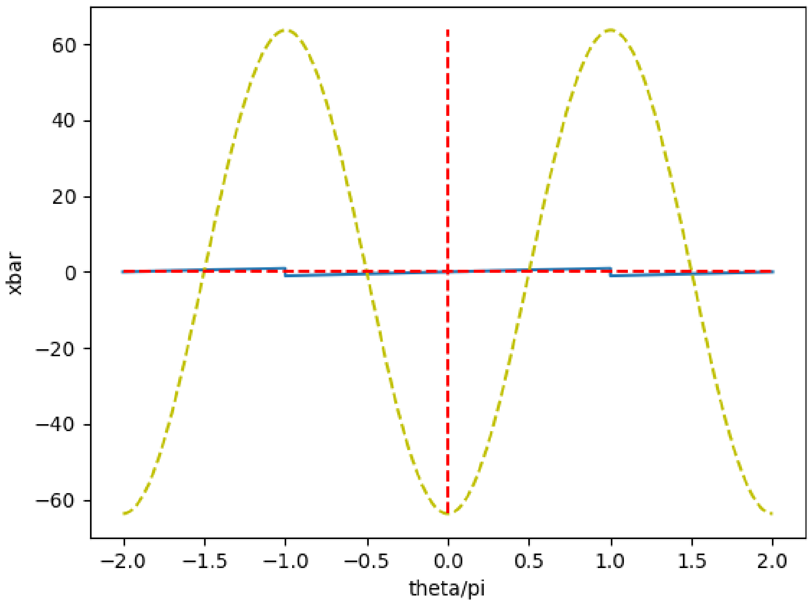
We can see that for resonence, ϕ will approach π rapidly and this is what we observe. This is consistent with our knowledge about the phase of response when a system is driven below, at, and above resonance.



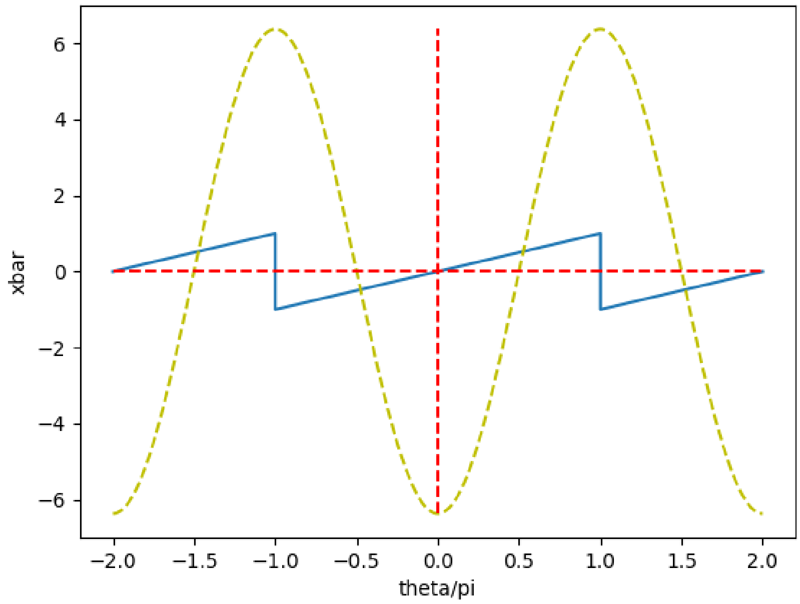
Now all that is left to do is play! Having python capable of graphing all this stuff is really great. It makes it so easy to try stuff.

First I will show what the problem explicitly asks for. It is very simple to have python show anything I want now that I set it up.

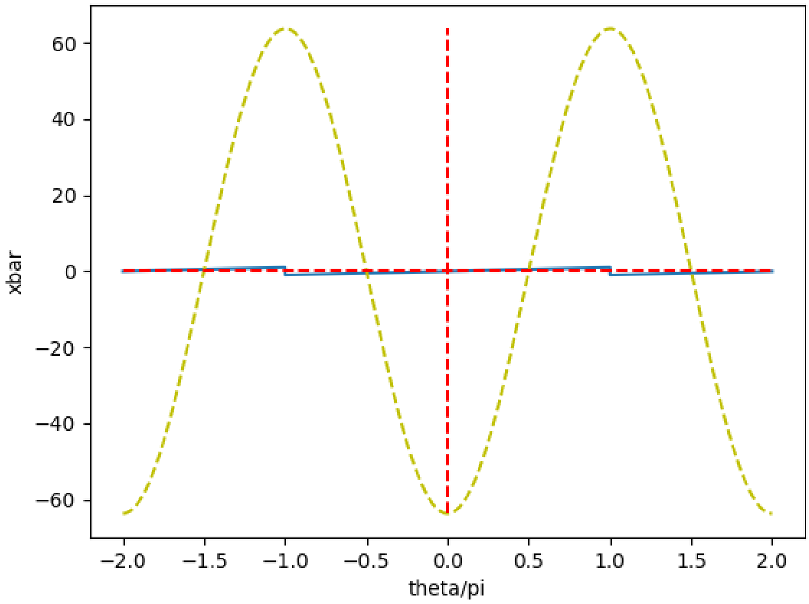
First is $\bar{\omega} = 1, \beta = .01, n = 6$



The response is off the charts! I looked for an error for a while and could not find one. Notice that the vertical axis is scaled to the problem, and it is responding with an amplitude of 60! It seems high, but it is reasonable that it respond so strongly because it is being driven at response. Also notice the textbook quarter cycle the drive has on the response. $\bar{\omega} = 1, \beta = .1, n = 1$

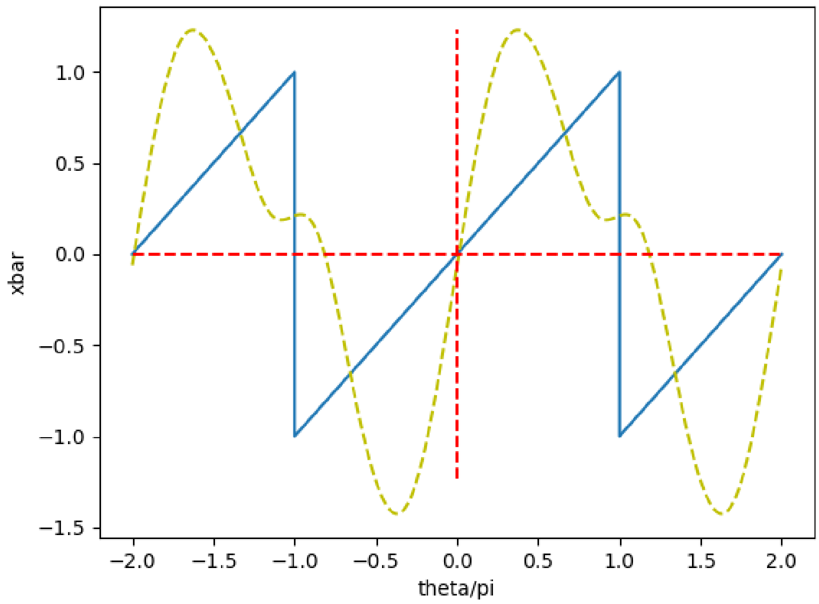


Notice that I can change n from 1 to as high as I want and this response doesn't change! This makes sense because the system selects out the resonant modes and does not respond much to the others. Here is $\bar{\omega} = 1, \beta = .01, n = 500$

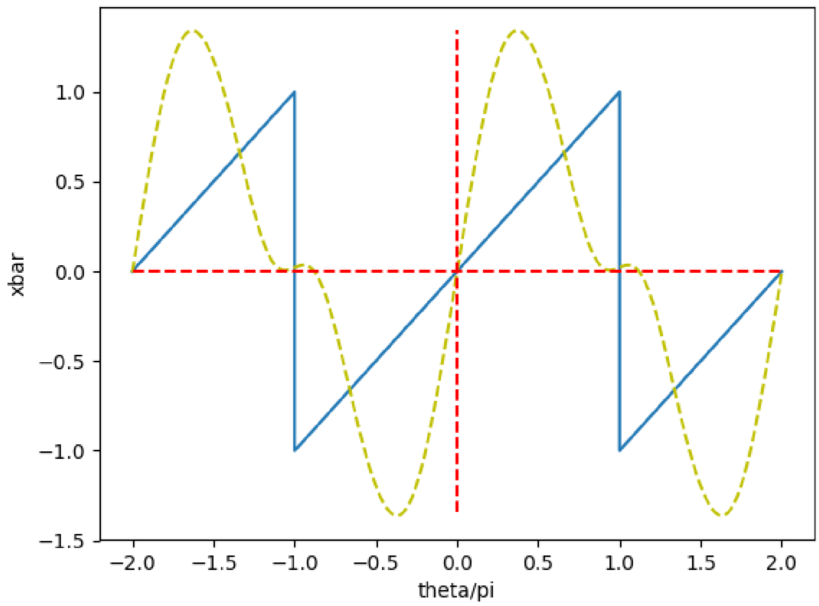


It is the same!

Next I'll change $\bar{\omega} = .66$. I expect it to be a little goofy and not respond very well because it is not being driven at the proper frequency.
$\bar{\omega} = .66, \beta = .1$



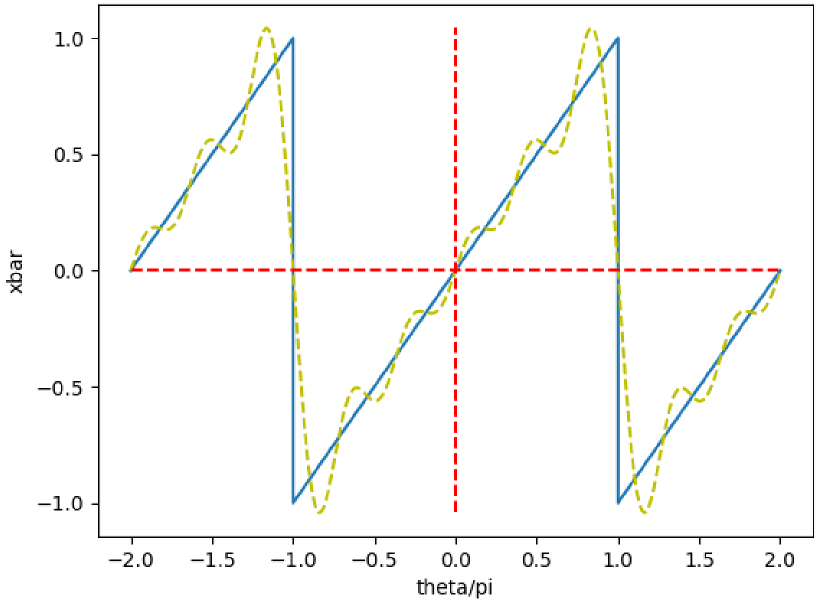
and for $\bar{\omega} = .66, \beta = .01$



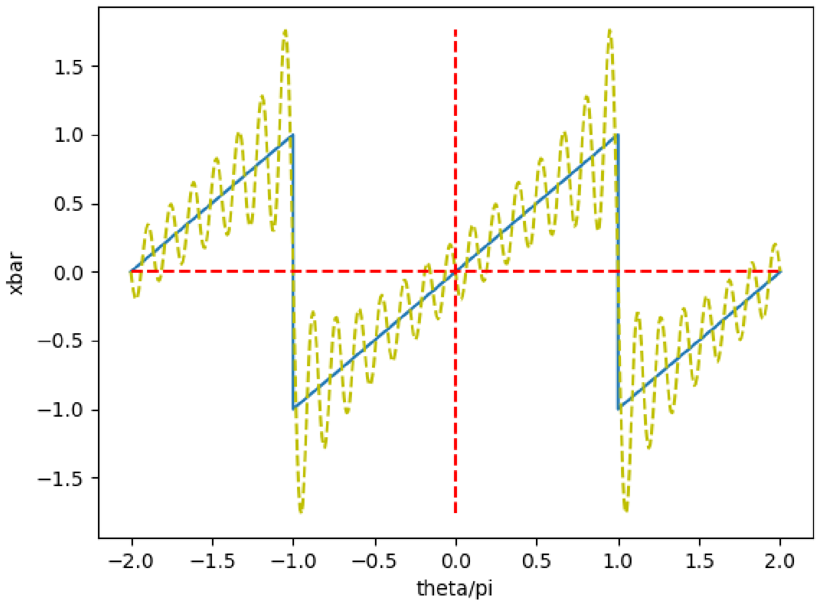
This is not what I expected! It looks like the max amplitude was not affected so much because the lack of resonant modes to grab on to was the limiting factor. There was just the small change in shape that I did not see coming.

3 Messing Around

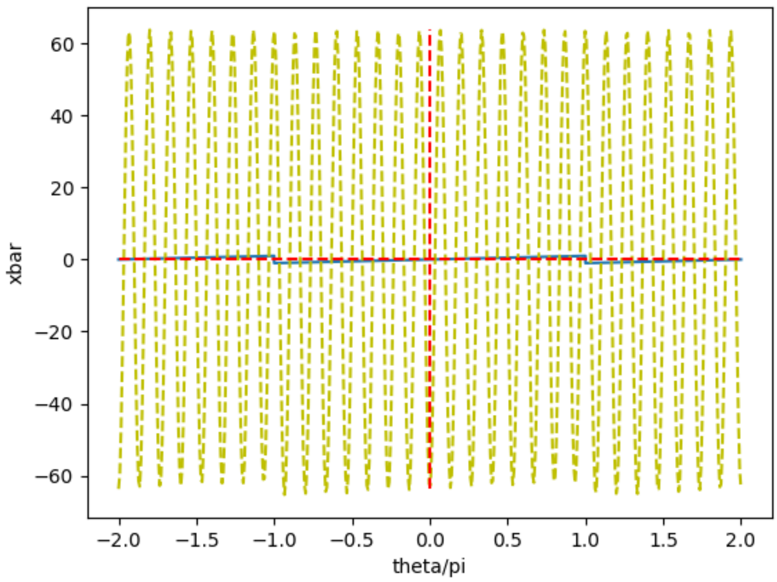
Some of the more interesting things I found! If you make $\bar{\omega}$ very small (I think of this as a very tight spring, but all that matters is the drive frequency is « the natural frequency), The response follows the approximate force very closely



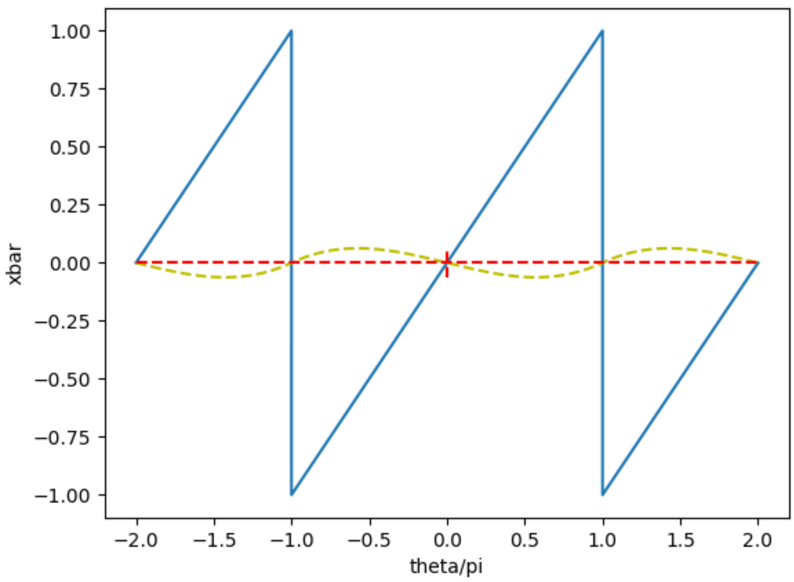
You can play with n and the tight spring will make it bounce around the force.



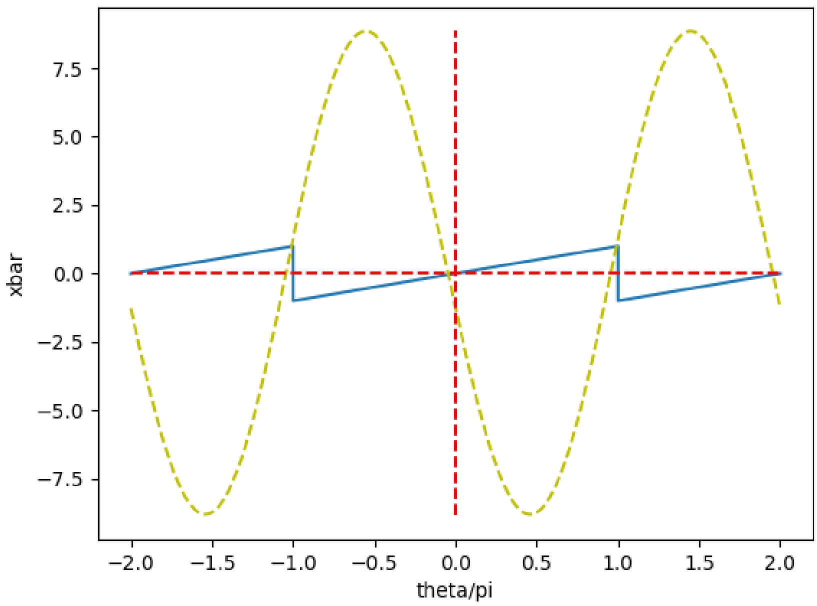
The trick is n must be kept low enough so that a resonant mode is not introduced. As soon as it is, that one will explode



This works the other way too. If $\bar{\omega}$ is large it does not respond. It is the same for any n because adding higher modes does not help when the lowest one is already too high!



If you let $\bar{\omega}$ be very close to 1, you get a cool phase shift.



This is because the mode that is selected has an $arctan$(close to integer) that doesn't slam the phase shift to $0, \frac{\pi}{2}, \pi$. There is a lot more playing to be done!